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LOCATION OF ZEROS OF A CLASS OF POLYNOMIALS

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ABSTRACT
In this paper we consider a certain class of polynomials with certain conditions on their coefficients and find regions containing all or some of their zeros.
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INTRODUCTION
A famous result on the location of zeros of a polynomial with positive real monotonically increasing coefficients is the following theorem known as the Enestrom-Kakeya Theorem [8,9]:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0 .$$

Then all the zeros of P(z) lie in $|z| \le 1$.

The above result has been generalized, refined and extended in various ways (e.g see[1]-[11]).
Very recently Gulzar [6] proved the following result:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j, \operatorname{Im}(a_j)=\beta_j$, $j=0,1,2,......,n$ such that for some $\lambda, 0 \le \lambda \le n-1$ and for some $k_1, k_2 \ge 1, o < \tau \le 1$,

$$k_1\alpha_n \ge k_2\alpha_{n-1} \ge ...... \ge \tau\alpha_\lambda$$

and

$$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0| .$$

Then all the zeros of P(z) lie in

$$\left|z+\frac{(k_1-1)\alpha_n}{a_n}-\frac{(k_2-1)\alpha_{n-1}}{a_n}\right| \le \frac{k_1\alpha_n+(k_2-1)|\alpha_{n-1}|+|\alpha_\lambda|+L-\tau(|\alpha_\lambda|+\alpha_\lambda)+2\sum_{j=0}^{n}|\beta_j|}{|a_n|} .$$

Taking a_j real i.e. $\beta_j=0, \forall j=0,1,2,......,n$ and $a_\lambda > 0$, we get the following result from Theorem B:

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with such that for some $\lambda, 0 \le \lambda \le n-1$ and for some $k_1, k_2 \ge 1, o < \tau \le 1$,

$$k_1 a_n \ge k_2 a_{n-1} \ge ...... \ge \tau a_\lambda > 0$$

and

$$L=|a_\lambda-a_{\lambda-1}|+|a_{\lambda-1}-a_{\lambda-2}|+......+|a_1-a_0|+|a_0|.$$

Then all the zeros of P(z) lie in

$$\left|z+\frac{(k_1-1)\alpha_n}{a_n}-\frac{(k_2-1)\alpha_{n-1}}{a_n}\right|\le\frac{k_1a_n+(k_2-1)a_{n-1}+a_\lambda+L-2\tau a_\lambda}{a_n}.$$

MAIN RESULTS

In this paper we prove the following results which not only generalize the above results but also many other already known results in the literature can be derived from them:

Theorem 1: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that for some $\lambda, 0\le\lambda\le n-1$ and for some $k_1,k_2\ge1, o<\tau\le1$,

$$k_1|a_n|\ge k_2|a_{n-1}|\ge......\ge\tau|a_\lambda|$$

and for some real α,β,

$$|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2},\ j=\lambda+1,\lambda+2,......,n.$$

Then all the zeros of P(z) lie in

$$\left|z+(k_1-1)-\frac{(k_2-1)a_{n-1}}{a_n}\right|\le\frac{1}{|a_n|}[k_1|a_n|(\cos\alpha+\sin\alpha)+k_2|a_{n-1}|(1+\sin\alpha)-|a_{n-1}|+|a_\lambda|$$
$$-\tau|a_\lambda|(1+\cos\alpha-\sin\alpha)+L+2\sin\alpha\sum_{j=\lambda+1}^{n-1}|a_j|],$$

where

$$L=|a_\lambda-a_{\lambda-1}|+|a_{\lambda-1}-a_{\lambda-2}|+......+|a_1-a_0|+|a_0|.$$

Remark 1: Taking $\alpha=\beta=0$ in Theorem 1, we get Theorem C.

Taking $\tau=1$, Theorem 1 gives the following result:

Corollary 1: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that for some $\lambda, 0\le\lambda\le n-1$; $k_1,k_2\ge1$, ,

$$k_1|a_n|\ge k_2|a_{n-1}|\ge......\ge|a_\lambda|$$

and for some real α,β,

$$|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2},\ j=\lambda+1,\lambda+2,......,n.$$

Then all the zeros of P(z) lie in

$$\left|z+\frac{(k_1-1)a_n}{a_n}-\frac{(k_2-1)a_{n-1}}{a_n}\right|\le\frac{1}{|a_n|}[k_1|a_n|(\cos\alpha+\sin\alpha)+k_2|a_{n-1}|(1+\sin\alpha)-|a_{n-1}|$$
$$-|a_\lambda|(\cos\alpha-\sin\alpha)+L+2\sum_{j=0}^{n}|\beta_j|],$$

where

$$L=|a_\lambda-a_{\lambda-1}|+|a_{\lambda-1}-a_{\lambda-2}|+......+|a_1-a_0|+|a_0|.$$

Taking $k_1=k_2=\tau=1$, Theorem 1 reduces to the following result :

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Corollary 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $\lambda, 0\le\lambda\le n-1$,

$$|a_n|\ge|a_{n-1}|\ge......\ge|a_\lambda|$$

and for some real α,β,

$$|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2},\ j=\lambda+1,\lambda+2,......,n.$$

Then all the zeros of P(z) lie in

$$|z|\le\frac{1}{|a_n|}[|a_n|(\cos\alpha+\sin\alpha)+|a_{n-1}|\sin\alpha-|a_\lambda|(\cos\alpha-\sin\alpha)+L+2\sum_{j=0}^{n}|\beta_j|].$$

where

$$L=|a_\lambda-a_{\lambda-1}|+|a_{\lambda-1}-a_{\lambda-2}|+......+|a_1-a_0|+|a_0|.$$

Taking $k_2=1$, Theorem 1 reduces to the following result :

Corollary 3: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $\lambda, 0\le\lambda\le n-1$ and for some $k_1\ge1, o<\tau\le1$,

$$k_1|a_n|\ge|a_{n-1}|\ge......\ge\tau|a_\lambda|$$

and for some real α,β,

$$|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2},\ j=\lambda+1,\lambda+2,......,n,$$

Then all the zeros of P(z) lie in

$$\left|z+\frac{(k_1-1)a_n}{a_n}\right|\le\frac{1}{|a_n|}[k_1|a_n|(\cos\alpha+\sin\alpha)+|a_{n-1}|\sin\alpha+|a_\lambda|$$

$$-\tau|a_\lambda|(1+\cos\alpha-\sin\alpha)+L+2\sum_{j=0}^{n}|\beta_j|],$$

where

$$L=|a_\lambda-a_{\lambda-1}|+|a_{\lambda-1}-a_{\lambda-2}|+......+|a_1-a_0|+|a_0|.$$

Next , we prove the following result:

Theorem 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $\lambda, 0\le\lambda\le n-1$, for some $k_1,k_2\ge1, o<\tau\le1$,

$$k_1|a_n|\ge k_2|a_{n-1}|\ge......\ge\tau|a_\lambda|$$

and for some real α,β,

$$|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2},\ j=\lambda+1,\lambda+2,......,n.$$

Then the number of zeros of P(z) in $|z|\le\delta, 0<\delta<1$ does not exceed

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CODEN: IJESS7

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{1}{|a_0|}[k_1|a_n|(1+\cos\alpha+\sin\alpha)+2(k_2-1)|a_{n-1}|)+|\alpha_\lambda|+L-\tau|a_\lambda|(\cos\alpha-\sin\alpha+1)$$

$$+2\sum_{j=\lambda+1}^{n-1}|a_j|],$$

where L is as given in Theorem 1.

For different choices of the parameters as in Theorem 1, we get many interesting results from Theorem 2 as well, which generalize many known results in the literature. For example taking $\tau=1$, we get the following result from Theorem 2:

Corollary 4: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that for some $\lambda, 0\le\lambda\le n-1$, for some $k_1,k_2\ge 1,$,

$$k_1|a_n|\ge k_2|a_{n-1}|\ge ......\ge|a_\lambda|$$

and for some real α,β,

$$|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}, j=\lambda+1,\lambda+2,......,n.$$

Then the number of zeros of P(z) in $|z|\le\delta, 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{1}{|a_0|}[k_1|a_n|(1+\cos\alpha+\sin\alpha)+2(k_2-1)|a_{n-1}|)+L-|a_\lambda|(\cos\alpha-\sin\alpha)+2\sum_{j=\lambda+1}^{n-1}|a_j|],$$

where L is as given in Theorem 1.

Taking $k_1=k, k_2=1$, Theorem 2 reduces to the following result :

Corollary 5: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that for some $\lambda, 0\le\lambda\le n-1$, for some $k_1\ge 1, o<\tau\le 1,$

$$k|a_n|\ge|a_{n-1}|\ge ......\ge\tau|a_\lambda|$$

and for some real α,β,

$$|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}, j=\lambda+1,\lambda+2,......,n.$$

Then the number of zeros of P(z) in $|z|\le\delta, 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{1}{|a_0|}[k|a_n|(1+\cos\alpha+\sin\alpha)+|\alpha_\lambda|+L-\tau|a_\lambda|(\cos\alpha-\sin\alpha+1)+2\sum_{j=\lambda+1}^{n-1}|a_j|],$$

where L is as given in Theorem 1.

Taking $k_1=k_2=\tau=1$, we get the following result from Theorem 2:

Corollary 6: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that for some $\lambda, 0\le\lambda\le n-1,$,

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$$|a_n| \geq |a_{n-1}| \geq ...... \geq |a_\lambda|$$

and for some real α, β,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = \lambda+1, \lambda+2, ..., n.$$

Then the number of zeros of P(z) in $|z| \leq \delta, 0 < \delta < 1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}} \log\frac{1}{|a_0|}[|a_n|(1+\cos\alpha+\sin\alpha) + L - |a_\lambda|(\cos\alpha - \sin\alpha) + 2\sum_{j=\lambda+1}^{n-1}|a_j|],$$

where L is as given in Theorem 1.

LEMMA

For the proofs of the above results we need the following lemma:

Lemma: For any two complex numbers b_1, b_2 such that $|b_1| \geq |b_2|$ and $|\arg b_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 1,2$ for some real α, β,

$$|b_1 - b_2| \leq (|b_1| - |b_2|)\cos\alpha + (|b_1| + |b_2|)\sin\alpha.$$

The above lemma is due to Govil and Rahman [4].

PROOFS OF THEOREMS

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda \\
&\quad + ... + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} - (k_1 - 1)a_n z^n + (k_1 a_n - k_2 a_{n-1})z^n + (k_2 - 1)a_{n-1}z^n + (k_2 a_{n-1} - a_{n-2})z^{n-1} \\
&\quad - (k_2 - 1)a_{n-1}z^{n-1} + (a_{n-2} - a_{n-3})z^{n-2} + ... + (a_{\lambda+1} - \tau a_\lambda)z^{\lambda+1} + (\tau - 1)a_\lambda z^{\lambda+1} \\
&\quad + (a_\lambda - a_{\lambda-1})z^\lambda + ... + (a_1 - a_0)z + a_0
\end{aligned}$$

For $|z| > 1$ so that $\frac{1}{|z|^j} < 1, \forall j = 1,2,......, n$, we have, by using the hypothesis

$$\begin{aligned}
|F(z)| &\geq |a_n z + (k_1 - 1)a_n - (k_2 - 1)a_{n-1}||z|^n - [|k_1 a_n - k_2 a_{n-1}||z|^n + |k_2 - 1||a_{n-1}||z|^{n-1} + |k_2 a_{n-1} - a_{n-2}|.|z|^{n-1} \\
&\quad + |a_{n-2} - a_{n-3}||z.|^{n-2} + ... + |a_{\lambda+1} - \tau a_\lambda||z|^{\lambda+1} + |\tau - 1||a_\lambda||z|^{\lambda+1} + |a_\lambda - a_{\lambda-1}||z|^\lambda \\
&\quad + ... + |a_1 - a_0||z| + |a_0| \\
&= |z|^n[|a_n z + (k_1 - 1)a_n - (k_2 - 1)a_{n-1}| - \{|k_1 a_n - k_2 a_{n-1}| + \frac{(k_2 - 1)|a_{n-1}|}{|z|} + \frac{|k_2 a_{n-1} - a_{n-2}|}{|z|} \\
&\quad + \frac{|a_{n-2} - a_{n-3}|}{|z|^2} + ... + \frac{|a_{\lambda+1} - \tau a_\lambda|}{|z|^{n-\lambda-1}} + \frac{(1-\tau)|a_\lambda|}{|z|^{n-\lambda-1}} + \frac{|a_\lambda - a_{\lambda-1}|}{|z|^{n-\lambda}}
\end{aligned}$$

$$+ ...... + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\}]$$

$$> |z|^n [|a_n z + (k_1 - 1)a_n - (k_2 - 1)a_{n-1}| - \{|k_1 a_n - k_2 a_{n-1}| + (k_2 - 1)|a_{n-1}|$$
$$+ |k_2 a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}| + ... + |a_{\lambda+1} - \tau a_\lambda| + (1-\tau)|a_\lambda|$$
$$+ |a_\lambda - a_{\lambda-1}| + ... + |a_1 - a_0| + |a_0|$$

$$\geq |z|^n [|a_n z + (k_1 - 1)a_n - (k_2 - 1)a_{n-1}| - \{(k_1|a_n| - k_2|a_{n-1}|)\cos\alpha + (k_1|a_n| + k_2|a_{n-1}|)\sin\alpha$$
$$+ (k_2 - 1)|a_{n-1}| + (k_2|a_{n-1}| - |a_{n-2}|)\cos\alpha + (k_2|a_{n-1}| + |a_{n-2}|)\sin\alpha$$
$$+ (|a_{n-2}| - |a_{n-3}|)\cos\alpha + (|a_{n-2}| + |a_{n-3}|)\sin\alpha + ... + (|a_{\lambda+1}| - \tau|a_\lambda|)\cos\alpha$$
$$+ (|a_{\lambda+1}| + \tau|a_\lambda|)\sin\alpha + (1-\tau)|a_\lambda| + |a_\lambda - a_{\lambda-1}| + ... + |a_1 - a_0| + |a_0|\}]$$

$$= |z|^n [|a_n z + (k_1 - 1)a_n - (k_2 - 1)a_{n-1}| - \{(k_1|a_n|(\cos\alpha + \sin\alpha) + (k_2 - 1)|a_{n-1}|$$
$$- \tau|a_\lambda|(\cos\alpha - \sin\alpha + 1) + |a_\lambda| + 2\sin\alpha \sum_{j=\lambda+1}^{n-1} |a_j|$$

>0

if

$$|a_n z + (k_1 - 1)a_n - (k_2 - 1)a_{n-1}| > (k_1|a_n|(\cos\alpha + \sin\alpha) + (k_2 - 1)|a_{n-1}|$$
$$- \tau|a_\lambda|(\cos\alpha - \sin\alpha + 1) + |a_\lambda| + 2\sin\alpha \sum_{j=\lambda+1}^{n-1} |a_j|.$$

This shows that those zeros of F(z) whose modulus is greater than 1 lie in

$$|a_n z + (k_1 - 1)a_n - (k_2 - 1)a_{n-1}| \leq (k_1|a_n|(\cos\alpha + \sin\alpha) + (k_2 - 1)|a_{n-1}|$$
$$- \tau|a_\lambda|(\cos\alpha - \sin\alpha + 1) + |a_\lambda| + L + 2\sin\alpha \sum_{j=\lambda+1}^{n-1} |a_j|$$

.

i.e.

$$\left| z + (k_1 - 1) - (k_2 - 1)\frac{a_{n-1}}{a_n} \right| \leq \frac{1}{|a_n|}[(k_1|a_n|(\cos\alpha + \sin\alpha) + (k_2 - 1)|a_{n-1}|$$
$$- \tau|a_\lambda|(\cos\alpha - \sin\alpha + 1) + |a_\lambda| + L + 2\sin\alpha \sum_{j=\lambda+1}^{n-1} |a_j|].$$

Since the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality and since the zeros of P(z) are also the zeros of F(z) , it follows that all the zeros of P(z) lie in

$$\left| z + (k_1 - 1) - (k_2 - 1)\frac{a_{n-1}}{a_n} \right| \leq \frac{1}{|a_n|}[(k_1|a_n|(\cos\alpha + \sin\alpha) + (k_2 - 1)|a_{n-1}|$$
$$- \tau|a_\lambda|(\cos\alpha - \sin\alpha + 1) + |a_\lambda| + 2\sin\alpha \sum_{j=\lambda+1}^{n-1} |a_j|].$$

That proves Theorem 1.

Proof of Theorem 2: Consider the polynomial

$$F(z) = (1 - z)P(z)$$
$$= (1 - z)(a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0)$$
$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda$$

$$+......+(a_1-a_0)z+a_0$$
$$=-a_nz^{n+1}-(k_1-1)a_nz^n+(k_1a_n-k_2a_{n-1})z^n+(k_2-1)a_{n-1}z^n+(k_2a_{n-1}-a_{n-2})z^{n-1}$$
$$-(k_2-1)a_{n-1}z^{n-1}+(a_{n-2}-a_{n-3})z^{n-2}+......+(a_{\lambda+1}-\tau a_\lambda)z^{\lambda+1}+(\tau-1)a_\lambda z^{\lambda+1}$$
$$+(a_\lambda-a_{\lambda-1})z^\lambda+......+(a_1-a_0)z+a_0$$

For $|z|\le 1$, we have, by using the hypothesis

$$|F(z)|\le|a_n|+(k_1-1)|a_n|+|k_1a_n-k_2a_{n-1}|+(k_2-1)|a_{n-1}|+|k_2a_{n-1}-a_{n-2}|+(k_2-1)|a_{n-1}|$$
$$+|a_{n-2}-a_{n-3}|+......+|a_{\lambda+1}-\tau a_\lambda|+(1-\tau)|a_\lambda|+|a_\lambda-a_{\lambda-1}|+......+|a_1-a_0|+|a_0|$$
$$\le|a_n|+(k_1-1)|a_n|+2(k_2-1)|a_{n-1}|+(k_1|a_n|-k_2|a_{n-1}|)\cos\alpha+(k_1|a_n|+k_2|a_{n-1}|)\sin\alpha$$
$$+(k_2|a_{n-1}|-|a_{n-2}|)\cos\alpha+(k_2|a_{n-1}|+|a_{n-2}|)\sin\alpha+(|a_{n-2}|-|a_{n-3}|)\cos\alpha$$
$$+(|a_{n-2}|+|a_{n-3}|)\sin\alpha+......+(|a_{\lambda+1}|-\tau|a_\lambda|)\cos\alpha+(|a_{\lambda+1}|+\tau|a_\lambda|)\sin\alpha$$
$$+(1-\tau)|a_\lambda|+|a_\lambda-a_{\lambda-1}|+......+|a_1-a_0|+|a_0|$$
$$=k_1|a_n|(1+\cos\alpha+\sin\alpha)+2(k_2-1)|a_{n-1}|)+|\alpha_\lambda|+L-\tau|\alpha_\lambda|(\cos\alpha-\sin\alpha+1)$$
$$+2\sin\alpha\sum_{j=\lambda+1}^{n-1}|a_j|.$$

Since F(z) is analytic for $|z|\le 1,\ F(0)=a_0\ne 0$, it follows by the Lemma that the number of zeros of F(z) in $|z|\le\delta, 0<\delta<1$ des not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{1}{|a_0|}[k_1|a_n|(1+\cos\alpha+\sin\alpha)+2(k_2-1)|a_{n-1}|)+|\alpha_\lambda|+L-\tau|a_\lambda|(\cos\alpha-\sin\alpha+1)$$
$$+2\sum_{j=\lambda+1}^{n-1}|a_j|].$$

Since the zeros of P(z) are also the zeros of F(z) , it follows that the number of zeros of P(z) in $|z|\le\delta, 0<\delta<1$ des not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{1}{|a_0|}[k_1|a_n|(1+\cos\alpha+\sin\alpha)+2(k_2-1)|a_{n-1}|)+|\alpha_\lambda|+L-\tau|a_\lambda|(\cos\alpha-\sin\alpha+1)$$
$$+2\sum_{j=\lambda+1}^{n-1}|a_j|].$$

That completes the proof of Theorem 2.

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